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Continuous valuations

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0 Introduction

In this paper we study, for a certain type of topological rings A, the topological space Cont A of all equivalence classes of continuous valuations of A. The space Cont A is defined as follows. Let v: $A \to \Gamma \cup \{0\}$ be a valuation of A, where Γ is an ordered multiplicative group generated by $\mathrm{im}(v) \setminus \{0\}$. On $\Gamma \cup \{0\}$ we introduce the topology such that $U \subseteq \Gamma \cup \{0\}$ is open iff $0 \notin U$ or $\{x \in \Gamma \mid x < \gamma\} \subseteq U$ for some $\gamma \in \Gamma$. We call v continuous if the mapping v: $A \to \Gamma \cup \{0\}$ is continuous with respect to the ring topology on A and the topology on $\Gamma \cup \{0\}$ just defined. Two continuous valuations v: $A \to \Gamma \cup \{0\}$ and w: $A \to \Delta \cup \{0\}$ are called equivalent if there exists an isomorphism f: $\Gamma \cup \{0\} \to \Delta \cup \{0\}$ of ordered monoids such that $w = f \circ v$. Then Cont A is the set of all equivalence classes of continuous valuations of A equipped with the topology generated by the sets $\{v \in \mathrm{Cont}\, A \mid v(a) \leqq v(b) \neq 0\}$ $(a, b \in A)$.

Our study of the topological spaces Cont A is motivated by the following result. Let A be a Tate algebra over a complete, non-archimedean, valued field [2, 4, 13]. One can associate with A a topological space X_A which is uniquely determined up to homeomorphism. Namely, let $\mathscr{T}_A$ be the Grothendieck topology of the rigid analytic variety Sp A associated with A [4, III.2.1]. Then it is easily seen that the topos $\mathrm{Shv}(\mathscr{T}_A)$ of $\mathscr{T}_A$ is spatial, i.e. there exists a sober topological space X_A such that $\mathrm{Shv}(\mathscr{T}_A)$ is equivalent to the topos $\mathrm{Shv}(X_A)$ of X_A. By [6, IV.4.2.4] X_A is uniquely determined up to homeomorphism. In this paper we will show that X_A is homeomorphic to the topological subspace $\mathrm{Spa}(A, A^\circ) = \{v \in \mathrm{Cont}\, A \mid v(a) \leqq 1$ for every $a \in A^\circ\}$ of Cont A (A° denotes the set of power bounded elements of A). We will even show that $\mathrm{Shv}(\mathscr{T}_A)$ is *canonically* equivalent to $\mathrm{Shv}(\mathrm{Spa}(A, A^\circ))$.

Having seen that, for Tate algebras A, the topological space Cont A occurs very naturally in rigid analytic geometry, one can ask for applications of Cont A for more general topological rings A. In this paper we restrict ourselves to a class of topological rings which I call f-adic rings: A topological ring is f-adic if it contains an open subring which is adic and has a finitely generated ideal of definition. Then Tate algebras and adic rings with finitely generated ideals of definition are f-adic. The main result of this paper is that, for every f-adic ring A, the topological space Cont A is spectral. Moreover, we will give a rather explicit description of the boolean algebra of constructible subsets of Cont A. We will see that there is a good notion of rational subsets of Cont A, similar to the rational subsets of affinoid rigid analytic varieties. Our main application of the topological spaces Cont A will be given in a second paper [8], where we will show that there exists a natural structure sheaf $\mathcal{O}_A$ on Cont A such that, glueing together locally ringed spaces of the form (Cont A, $\mathcal{O}_A$), one obtains a category of locally ringed spaces which generalizes both the category of rigid analytic varieties and the category of locally noetherian formal schemes.

The paper is organized as follows. In §1 we discuss the notion of a f-adic ring. In §2 we consider Cont A in the special case that the topology of A is discrete (every discrete topological ring is f-adic), and in §3 we study Cont A for arbitrary f-adic rings A. In §4 we prove the result mentioned above that $\mathrm{Shv}(\mathcal{T}_A)$ is canonically equivalent to $\mathrm{Shv}(\mathrm{Spa}(A, A^\circ))$ for every Tate algebra A.

1 F-adic rings

All rings are tacitly assumed to be commutative with unit element.

We recall some notations. Let A be a topological ring. A subset B of A is called bounded if, for every neighbourhood U of 0 in A, there exists a neighbourhood V of 0 in A with $v \cdot b \in U$ for every $v \in V$, $b \in B$. An element a of A is called power-bounded if the set $\{a^n | n \in \mathbb{N}\}$ is bounded. A° denotes the set of all power-bounded elements of A. An element a of A is called topologically nilpotent if $(a^n | n \in \mathbb{N})$ is a zero sequence. $A^{\circ\circ}$ denotes the set of all topologically nilpotent elements of A. The ring A is called adic if there exists an ideal I of A such that $\{I^n | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0, and such an ideal is called an ideal of definition of A. For subsets S and T of A, let $S \cdot T$ be the subgroup of A generated by the elements $s \cdot t$ with $s \in S$ and $t \in T$.

Definition. (i) A topological ring A is called *f-adic* if there exist a subset U of A and a finite subset T of U such that $\{U^n | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in A and $T \cdot U = U^2 \subseteq U$.

(ii) A topological ring A is called *Tate* (or A is called a *Tate ring*) if A is f-adic and has a topologically nilpotent unit.

Examples 1.1 (i) Every adic ring with a finitely generated ideal of definition is f-adic.

(ii) Let A be a ring and I a finitely generated ideal of A. We equip the polynomial ring $A[X]$ with the group topology such that $\{U_n | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 where $U_n := \{\sum a_k X^k \in A[X] | a_k \in I^{n+k}$ for all $k\}$. Then $A[X]$ is a f-adic ring. But $A[X]$ is not an adic ring if $I^m \neq I^{m+1}$ for every $m \in \mathbb{N}$.

(iii) Let $(k, |\ |)$ be a non-trivial non-archimedean value field and $(A, \|\ \|)$ a normed algebra over $(k, |\ |)$ [2, 3.1.1]. Then A equipped with the topology induced by $\|\ \|$ is a Tate ring. (Indeed, if $A_0 := \{a \in A | \|a\| \leq 1\}$ and r is an element of k with $0 < |r| < 1$ then $\{r^n \cdot A_0 | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in A.)

(iv) Let B be a ring, s an element of B and $\varphi: B \to B_s$ the localization of B by s. We equip B_s with the group topology such that $\{\varphi(s^n B) | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0. Then it is easily seen that B_s is a Tate ring.

Definition. A subring A_0 of a f-adic ring A is called a *ring of definition* of A if A_0 is open in A and if the subspace topology of A on A_0 is adic.

Proposition 1. *Let A be a* f*-adic ring. Then*

(i) *A has a ring of definition.*

(ii) *A subring A_0 of A is a ring of definition of A if and only if A_0 is open and bounded in A.*

(iii) *Every ring of definition of A has a finitely generated ideal of definition.*

Proof. Obviously, an open and adic subring of A is bounded. Let U be a subset of A and T a finite subset of U such that $T \cdot U = U^2 \subseteq U$ and $\{U^n | n \in \mathbb{N}\}$ is a fundamental system of neighourhoods of 0. For every $n \in \mathbb{N}$ put $T(n) = \{t_1 \cdot \ldots \cdot t_n | t_1, \ldots, t_n \in T\}$. Let A_0 be an open and bounded subring of A. Choose a $k \in \mathbb{N}$ with $T(k) \subseteq A_0$, and put $I = T(k) \cdot A_0$. In order to prove (ii) and (iii) we show that $\{I^n | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0. Let ℓ be a natural number with $U^\ell \subseteq A_0$. Then we have for every $n \in \mathbb{N}$, $I^n = T(nk) \cdot A_0 \supseteq T(nk) \cdot U^\ell = U^{\ell + nk}$. Hence I^n is a neighbourhood of 0. Let V be a neighbourhood of 0. Since A_0 is bounded, there exists a $m \in \mathbb{N}$ with $U^m \cdot A_0 \subseteq V$. Then $I^m \subseteq V$.

Let W be the subgroup of A generated by U. Then $\mathbb{Z} \cdot 1 + W$ is an open and bounded subring of A. Hence A has a ring of definition.

Corollary 1.3 *Let A be a* f*-adic ring and A° the set of all power-bounded elements of A. Then*

(i) *If A_0 and A_1 are rings of definition of A then also $A_0 \cap A_1$ and $A_0 \cdot A_1$ are rings of definition of A.*

(ii) *Let B be a bounded subring of A and C an open subring of A with $B \subseteq C$. Then there exists a ring of definition A_0 of A with $B \subseteq A_0 \subseteq C$.*

(iii) *A° is a subring of A and A° is the union of all rings of definition of A.*

Corollary 1.4 (i) *An adic ring is* f*-adic if and only if it has a finitely generated ideal of definition.*

(ii) *A* f*-adic ring is adic if and only if it is bounded.*

(iii) *Let A be a topological ring and B an open subring of A. Then A is* f*-adic if and only if B is* f*-adic.*

The following proposition says that every Tate ring is of the form described in (1.1.iv).

Proposition 1.5 *Let A be a Tate ring and B a ring of definition of A. Then*

(i) *B contains a topologically nilpotent unit of A.*

(ii) *Lets $s \in B$ be a topologically nilpotent unit of A. Then $A = B_s$ and sB is an ideal of definition of B.*

Proof. (i) Let t be a topologically nilpotent unit of A. Then $t^n \in B$ for some $n \in \mathbb{N}$.

(ii) For every $a \in A$ there exists a $n \in \mathbb{N}$ with $s^n a \in B$. Hence $A = B_s$. For every $n \in \mathbb{N}$, the mapping $A \to A$, $a \mapsto s^n a$ is a homeomorphism of A. Hence $s^n B$ is open. Since B is bounded, there exists, for every neighbourhood V of 0, a $n \in \mathbb{N}$ with $s^n B \subseteq V$.

In this note *complete* always means *hausdorff and complete.*

Lemma 1.6 *Let A be a* f*-adic ring, B a ring of definition of A and I an ideal of definition of B. Let $\hat{A}$ and $\hat{B}$ be the completions of A and B. We consider $\hat{B}$ as an open subring of $\hat{A}$. Then*

(i) *$\hat{A}$ is* f*-adic, $\hat{B}$ is a ring of definition of $\hat{A}$ and $I \cdot \hat{B}$ is an ideal of definition of $\hat{B}$.*

(ii) *The canonical diagram*

$$\begin{tikzcd}
\hat{A} & A \arrow[l] \\
\hat{B} \arrow[u] & B \arrow[l] \arrow[u]
\end{tikzcd}$$

is cocartesian in the category of rings.

Proof. (i) By [3, III.2.12 Corollary 2] $\hat{B}$ is adic with ideal of definition $I \cdot \hat{B}$. Hence $\hat{A}$ is f-adic.

(ii) We consider the canonical commutative diagram

$$\begin{tikzcd}
\hat{A} & & A \arrow[ll, "i"'] \arrow[dl, "f"] \\
& \hat{B} \otimes_B A \arrow[ul, "j"] & \\
\hat{B} \arrow[uu] \arrow[ur, "g"] & & B. \arrow[ll] \arrow[uu]
\end{tikzcd}$$

We have to show that j is an isomorphism. For every $\hat{a} \in \hat{A}$ we choose $a \in A$ and $\hat{b} \in \hat{B}$ with $\hat{a} = i(a) + \hat{b}$, and put $h(\hat{a}) := f(a) + g(\hat{b}) \in \hat{B} \otimes_B A$. This definition of $h(\hat{a})$ is independent of the representation $\hat{a} = i(a) + \hat{b}$, since $i^{-1}(\hat{B}) = B$. Hence the mapping $h: \hat{A} \to \hat{B} \otimes_B A$ has the properties

(1) h is additive

(2) $f = h \circ i$

(3) $g = h \mid \hat{B}$.

Next we show that h is a ring homomorphism. For that it suffices to show that there exists a complete ring topology on $\hat{B} \otimes_B A$ such that f and h are continuous (by (2)). In order to construct this topology we remark that g is injective (since $j \circ g$ is injective). We equip $\hat{B} \otimes_B A$ with the group topology such that g is an open embedding. Then by (3) h is continuous at 0. Also f is continuous at 0. Since h and f are additive, we obtain that h and f are continuous. It remains to show that $\hat{B} \otimes_B A$ is a topological ring. Let $\hat{b} \otimes a$ be an element of $\hat{B}\otimes_B A$ and U a neighbourhood of 0 in $\hat{B}\otimes_B A$. We have to show that there exists a neighbourhood V of 0 in $\hat{B}\otimes_B A$ with $(\hat{b}\otimes a)\cdot V\subseteq U$. Let S be an open ideal of $\hat{B}$ with $g(S)\subseteq U$, and let T be a neighbourhood of 0 in B with $T\cdot a\subseteq i^{-1}(S)\subseteq B$. Then $V:=g(i(T)\cdot\hat{B})$ is a neighbourhood of 0 in $\hat{B}\otimes_B A$ with $(\hat{b}\otimes a)\cdot V=(\hat{b}\otimes a)\cdot g(i(T))\cdot g(\hat{B})=(\hat{b}\otimes a)\cdot f(T)\cdot g(\hat{B})=(\hat{b}\otimes Ta)\cdot g(\hat{B})$ $=(\hat{b}\cdot i(Ta)\otimes 1)\cdot g(\hat{B})\subseteq(S\otimes 1)\cdot g(\hat{B})=g(S)\subseteq U$. Thus we have proved that h is a ring homomorphism.

We have $j\circ h=\mathrm{id}_{\hat{A}}$ (by construction of h). Furthermore, $(h\circ j)(x)=x$ for all $x\in f(A)\cup g(\hat{B})$ (by (2) and (3)). Since $h\circ j$ is a ring homomorphism, we obtain that $h\circ j$ is the identity of $\hat{B}\otimes_B A$. Hence j is an isomorphism.

Corollary 1.7 *Let A be a* f*-adic ring and $\hat{A}$ the completion of A.*

(i) *If A has a noetherian ring of definition then the canonical ring homomorphism $A\to\hat{A}$ is flat.*

(ii) *If A is finitely generated over a noetherian ring of definition then $\hat{A}$ is noetherian.*

Let A and B be f-adic rings. Analogously to the adic situation one can define adic ring homomorphisms and ring homomorphisms of topologically finite type from A to B. (One has {continuous ring homomorphism $A\to B$} $\supseteq$ {adic ring homomorphism $A\to B$} $\supseteq$ {ring homomorphism of topologically finite type $A\to B$}.) In this note we need only adic ring homomorphisms: A ring homomorphism f: $A\to B$ is called *adic* if there exist rings of definition A_0, B_0 of A, B and an ideal of definition I of A_0 such that $f(A_0)\subseteq B_0$ and $f(I)\cdot B_0$ is an ideal of definition of B_0.

Lemma 1.8 *Let f: $A\to B$ be an adic ring homomorphism between* f*-adic rings. Then*

(i) *f is bounded (i.e. if $T\subseteq A$ is bounded in A then $f(T)$ is bounded in B).*

(ii) *If A_0, B_0 are rings of definition of A, B with $f(A_0)\subseteq B_0$ and if I is an ideal of definition of A_0 then $f(I)\cdot B_0$ is an ideal of definition of B_0.*

(iii) *To every ring of definition A_0 of A and every open subring B' of B with $f(A_0)\subseteq B'$ there exists a ring of definition B_0 of B with $f(A_0)\subseteq B_0\subseteq B'$.*

Proof. (i) obvious.

(ii) Let A_1, B_1 be rings of definition of A, B and let J, K be ideals of definition of A_1, B_1 with $f(A_1)\subseteq B_1$ and $f(J)\cdot B_1=K$. Put $A_2=A_0\cdot A_1$ and $B_2=B_0\cdot B_1$. Then A_2, B_2 are rings of definition of A, B with $f(A_2)\subseteq B_2$. Now $I\cdot A_2$ and $J\cdot A_2$ are ideals of definition of A_2, and $K\cdot B_2$ is an ideal of definition of B_2. Since $f(J\cdot A_2)\cdot B_2=K\cdot B_2$, we obtain that $L:=f(I\cdot A_2)\cdot B_2$ is an ideal of definition of B_2. We have $L=(f(I)\cdot B_0)\cdot B_2$, and hence $f(I)\cdot B_0$ is an ideal of definition of B_0.

(iii) follows from (i) and (1.3.ii).

Corollary 1.9 *Let f: $A\to B$ and g: $B\to C$ be ring homomorphism between* f*-adic rings.*

(i) *If f and g are adic then $g\circ f$ is adic.*

(ii) *If f and g are continuous and if $g\circ f$ is adic then g is adic.*

(iii) *Let A', B' be open subrings of A, B with $f(A')\subseteq B'$ and let f': $A'\to B'$ be the restriction of f. Then f is adic if and only if f' is adic.*

Proposition 1.10 *Let $f: A \to B$ be a continuous ring homomorphism between* f*-adic rings. Assume that A is a Tate ring. Then B is a Tate ring and f is adic.*

Proof. We use (1.5). Let A_0 be a ring of definition of A and let $s \in A_0$ be a topologically nilpotent unit of A. Then $f(s)$ is a topologically nilpotent unit of B, and hence B is a Tate ring. By (1.3.iii) there exists a ring of definition B_1 of B with $f(s) \in B_1$. Then $f(s)\, B_1$ is an ideal of definition of B_1. The set $f(A_0)$ is bounded in B, since there exists a $n \in \mathbb{N}$ with $f(s)^n f(A_0) = f(s^n A_0) \subseteq B_1$. Hence by (1.3. ii) there exists a ring of definition B_0 of B with $f(A_0) \subseteq B_0$. We know that sA_0 and $f(s)\, B_0$ are ideals of definition of A_0 and B_0. Hence f is adic.

2 Valuation spectrum

First we recall some notations about spectral spaces. A topological space X is called spectral if X is quasi-compact and has a basis of quasi-compact open sets stable under finite intersections and if every irreducible closed subset is the closure of a unique point [7]. For example, if A is a ring then the Zariski-spectrum $\operatorname{Spec} A$ is a spectral space. It can be shown (but will not be used here) that every spectral space is homeomorphic to the Zariski-spectrum of some ring [7]. Let X be a spectral space. A subset T of X is constructible [5, 0.2.3.10] if and only if T is contained in the boolean algebra of subsets of X generated by the quasi-compact open subsets, and a subset T of X is pro-constructible [5, I.7.2.2] if and only if T is an intersection of constructible subsets. A point $x \in X$ is called a specialization of a point $y \in X$ or y is called a generalization of x if x is contained in the closure of $\{y\}$ in X. A point $x \in X$ which has no proper generalization is called maximal. The topology of X generated by the constructible subsets is called the constructible topology, and X together with the constructible topology is denoted by X_{cons}. A mapping $f: X \to Y$ between spectral spaces is called spectral if $f: X \to Y$ and $f: X_{\text{cons}} \to Y_{\text{cons}}$ are continuous. In the following remark we note some important properties of spectral spaces (cf. [7]).

Remark 2.1 Let X be a spectral space and T a pro-constructible subset of X. Then

(i) T is quasi-compact in the topology of X and in the topology of X_{cons}. An open subset of X is constructible if and only if it is quasi-compact.

(ii) T is constructible iff $X \setminus T$ is pro-constructible

(iii) The closure of T in X is the set of the specializations of the points of T.

(iv) T (equipped with the subspace topology of X) is a spectral space and the inclusion $T \to X$ is spectral. A subset U of T is constructible (resp. constructible and open) iff there exists a constructible (resp. constructible and open) subset V of X with $U = V \cap T$.

(v) Let Y be a spectral space and $f: X \to Y$ be a continuous mapping. Then f is spectral iff the preimage of an open quasi-compact subset is quasi-compact iff the preimage of a constructible subset is constructible iff the preimage of a pro-constructible subset is pro-constructible. If f is spectral then the image of a pro-constructible subset is pro-constructible and if, moreover, f is surjective then $S \subseteq Y$ is constructible (resp. pro-constructible) iff $f^{-1}(S) \subseteq X$ is constructible (resp. pro-constructible).

(vi) Let Z be a set, $\mathscr{S}$ a quasi-compact topology on Z and $\mathscr{L}$ the set of open and closed subsets of $(Z, \mathscr{S})$. Let $\mathscr{T}$ be a T_0-topology of Z generated by a subset of $\mathscr{L}$. Then $(Z, \mathscr{T})$ is a spectral space and $\mathscr{L}$ is the set of constuctible subsets of $(Z, \mathscr{T})$.

The Zariski-spectrum, $A \mapsto \operatorname{Spec} A$, is a contravariant functor from the category of rings to the category of spectral spaces. Also the real spectrum, $A \mapsto \operatorname{Spec}_R A$, is a contravariant further from the category of rings to the category of spectral spaces [1, 7.1.17]. Similarly, using valuations of rings we will construct in this paragraph a contravariant functor from the category of rings to the category of spectral spaces which we call the valuation spectrum.

We begin with the definition of a valuation of a ring A. Let Γ be a totally ordered commutative group written multiplicatively. We add an element 0 to Γ and extend the multiplication and the ordering of Γ to $\Gamma \cup \{0\}$ by $\alpha \cdot 0 = 0 \cdot \alpha = 0$ and $0 \leqq \alpha$ for all $\alpha \in \Gamma \cup \{0\}$.

Definition [3, VI.3.1] A *valuation* of A with values in $\Gamma \cup \{0\}$ is a mapping $v\colon A \to \Gamma \cup \{0\}$ such that

(i) $v(x+y) \leqq \max\{v(x), v(y)\}$ for all $x, y \in A$

(ii) $v(x \cdot y) = v(x) \cdot v(y)$ for all $x, y \in A$

(iii) $v(0) = 0$ and $v(1) = 1$.

Let $v\colon \Gamma \cup \{0\}$ be a valuation. The subgroup of Γ generated by $\operatorname{im}(v) \setminus \{0\}$ is called the *value group* of v and is denoted by Γ_v. The convex subgroup of Γ generated by $\{v(a) \mid a \in A,\ v(a) \geqq 1\}$ is called the *characteristic subgroup* of v and is denoted by $c\Gamma$. The set $\operatorname{supp}(v) := v^{-1}(0)$ is a prime ideal of A and is called the support of v. The valuation v factorizes uniquely in $A \xrightarrow{g} qf(A/\operatorname{supp}(v)) \xrightarrow{\bar{v}} \Gamma \cup \{0\}$, where g is the canonical mapping and $\bar{v}$ is a valuation of the quotient field $K = qf(A/\operatorname{supp}(v))$ of $A/\operatorname{supp}(v)$. The valuation ring of $\bar{v}$ is denoted by $A(v)$, i.e. $A(v) = \{x \in K \mid \bar{v}(x) \leqq 1\}$.

Two valuations v and w of A are called *equivalent* if the following equivalent conditions are satisfied

(i) There is an isomorphism of ordered monoids $f\colon \Gamma_v \cup \{0\} \to \Gamma_w \cup \{0\}$ with $w = f \circ v$.

(ii) $\operatorname{supp}(v) = \operatorname{supp}(w)$ and $A(v) = A(w)$.

(iii) For all $a, b \in A$, $v(a) \geqq v(b)$ iff $w(a) \geqq w(b)$.

Let $S(A)$ denote the set of equivalence classes of valuations of A. In the following we often do not distinguish between a valuation and its equivalence class. Let T be the topology of $S(A)$ generated by the sets of the form $\{v \in S(A) \mid v(a) \leqq v(b) \neq 0\}$ with $a, b \in A$. We call the topological space $\operatorname{Spv} A := (S(A), T)$ the *valuation spectrum* of A.

Proposition 2.2 *Spv A is a spectral space. The boolean algebra of constructible subsets of* Spv A *is generated by the sets of form* $\{v \in \operatorname{Spv} A \mid v(a) \leqq v(b)\}$ *with* $a, b \in A$.

Proof. Every valuation v of A defines a binary relation $|_v$ on A by

$$a \,|_v\, b :\Leftrightarrow v(a) \geqq v(b).$$

Two valuations v and w of A are equivalent if and only if $|_v = |_w$. Therefore we have an injective mapping φ: $S(A) \to \mathscr{P}(A \times A)$, $v \mapsto |_v$. ($\mathscr{P}(A \times A)$ denotes the power set of $A \times A$.) We equip $\{0, 1\}$ with the discrete topology and $\mathscr{P}(A \times A) = \{0, 1\}^{A \times A}$ with the product topology. Then $\mathscr{P}(A \times A)$ is a compact Hausdorff space. The image $\operatorname{im}(\varphi)$ of φ is closed in $\mathscr{P}(A \times A)$ since $\operatorname{im}(\varphi)$ is the set of all binary relations $|$ on A which satisfy for all $a, b, c \in A$ the following conditions.

(1) $a|b$ or $b|a$.
(2) If $a|b$ and $b|c$ then $a|c$.
(3) If $a|b$ and $a|c$ then $a|b+c$.
(4) If $a|b$ then $ac|bc$.
(5) If $ac|bc$ and $0 \nmid c$ then $a|b$.
(6) $0 \nmid 1$.

Let T_1 be the topology of $S(A)$ such that φ: $(S(A), T_1) \to \mathscr{P}(A \times A)$ is a topological embedding and let K be the boolean algebra of subsets of $S(A)$ generated by the sets of the form $\{v \in S(A) \mid v(a) \leq v(b)\}$ with $a, b \in A$. Then $(S(A), T_1)$ is compact and K is the set of open and closed subsets of $(S(A), T_1)$. Now (2.2) follows from (2.1.vi).

Let f: $A \to B$ be a ring homomorphism and v: $B \to \Gamma \cup \{0\}$ a valuation of B. Then $v|A = \operatorname{Spv}(f)(v) := v \circ f$ is a valuation of A. The mapping $\operatorname{Spv}(f)$: $\operatorname{Spv} B \to \operatorname{Spv} A$ is spectral. So we have a contravariant functor Spv from the category of rings to the category of spectral spaces.

The specializations of the valuation spectrum $\operatorname{Spv} A$ of a ring A can be described as follows: Let v: $A \to \Gamma_v \cup \{0\}$ be a valuation of A. To every convex subgroup H of Γ_v we have the mappings

$$v/H: A \to \Gamma_v/H \cup \{0\}, \qquad a \mapsto \begin{cases} v(a) \bmod H & \text{if } v(a) \neq 0 \\ 0 & \text{if } v(a) = 0 \end{cases}$$

$$v|H: A \to H \cup \{0\}, \qquad a \mapsto \begin{cases} v(a) & \text{if } v(a) \in H \\ 0 & \text{if } v(a) \notin H. \end{cases}$$

It is easily seen that

(i) v/H is a valuation of A, and v/H is a generalization of v in $\operatorname{Spv} A$.

(ii) $v|H$ is a valuation of A iff $c\Gamma_v \subseteq H$, and in that case $v|H$ is a specialization of v in $\operatorname{Spv} A$.

A valuation w of A is called a *primary specialization* of v if there exists a convex subgroup H of Γ_v such that $c\Gamma_v \subseteq H$ and $w = v|H$, and a valuation w of A is called a *secondary specialization* of v if there exists a convex subgroup H of Γ_w with $v = w/H$. One can show [10, 1.2.4] that every specialization of v is a combination of a primary and a secondary specialization i.e.

Lemma 2.3 *Every specialization of v in* Spv *A is a primary specialization of a secondary specialization of v.*

In the next paragraph we will show that, for a f-adic ring A, the subspace of $\operatorname{Spv} A$ consisting of all continuous valuations of A is spectral. For that we need some preparations which we present here.

Let A be a ring and I an ideal of A such that there exists a finitely generated ideal J of A with $\sqrt{I}=\sqrt{J}$. Let $v\colon A\to\Gamma\cup\{0\}$ be a valuation of A. We say that an element γ of $\Gamma\cup\{0\}$ is *cofinal* in a subgroup H of Γ if for every $h\in H$ there exists a $n\in\mathbb{N}$ with $\gamma^n<h$.

Lemma 2.4 *If $v(I)\cap c\Gamma=\varnothing$ then there exists a greatest convex subgroup H of Γ such that $v(i)$ is cofinal in H for every $i\in I$. If furthermore $v(I)\neq\{0\}$ then $v(I)\cap H\neq\varnothing$.*

Proof. If $v(I)=\{0\}$ then of course $H=\Gamma$. Now assume $v(I)\neq\{0\}$. Let T be a finite set of generators of J and let H be the convex subgroup of Γ generated by $h:=\max\{v(t)\,|\,t\in T\}$. Then $v(i)$ is cofinal in H for every $i\in I$ (since $h<c\Gamma$) and H is the greatest convex subgroup of Γ with this property.

With (2.4) we can define for every valuation v of A

$$c\Gamma_v(I)=\begin{cases} c\Gamma_v & \text{if } v(I)\cap c\Gamma_v\neq\varnothing\\ \text{greatest convex} & \text{if } v(I)\cap c\Gamma_v=\varnothing\\ \text{subgroup } H \text{ of } \Gamma_v & \\ \text{such that } v(i) \text{ is} & \\ \text{cofinal in } H \text{ for} & \\ \text{every } i\in I. & \end{cases}$$

Then we have

Lemma 2.5 *For every valuation v of A the following conditions are equivalent.*

(i) $\Gamma_v=c\Gamma_v(I)$.

(ii) *$\Gamma_v=c\Gamma_v$ or $v(i)$ is cofinal in Γ_v for every $i\in I$.*

(iii) *$\Gamma_v=c\Gamma_v$ or $v(i)$ is cofinal in Γ_v for every element i of a set of generators of I.*

Proof. The equivalence of (i) and (ii) follows from the definition of $c\Gamma_v(I)$, and the equivalence of (ii) and (iii) follows from the fact that if $\Gamma_v\neq c\Gamma_v$ then $\{a\in A\,|\,v(a)$ is cofinal in $\Gamma_v\}$ is an ideal of A.

We put $\operatorname{Spv}(A,I)=\{v\in\operatorname{Spv}A\,|\,\Gamma_v=c\Gamma_v(I)\}$ and equip $\operatorname{Spv}(A,I)$ with the subspace topology of $\operatorname{Spv}A$. For every $v\in\operatorname{Spv}A$ we put $r(v)=v|c\Gamma_v(I)$. Then $r(v)\in\operatorname{Spv}(A,I)$, and $r(v)=v$ for every $v\in\operatorname{Spv}(A,I)$. Thus we have a retraction $r\colon\operatorname{Spv}A\to\operatorname{Spv}(A,I)$. Let $\mathcal{R}$ be the set of all subsets U of $\operatorname{Spv}(A,I)$ of the form

$$U=\{v\in\operatorname{Spv}(A,I)\,|\,v(f_1)\leqq v(g)\neq 0,\dots,v(f_n)\leqq v(g)\neq 0\},$$
$$\text{where } g,f_1,\dots,f_n \text{ are elements of } A$$
$$\text{with } I\subseteq\sqrt{(f_1,\dots,f_n)}.$$

Then we have

Proposition 2.6 (i) $\operatorname{Spv}(A,I)$ *is a spectral space.*

(ii) *$\mathcal{R}$ is a basis of the topology of* $\operatorname{Spv}(A,I)$ *that is closed under finite intersections, and every element of $\mathcal{R}$ is constructible in* $\operatorname{Spv}(A,I)$.

(iii) *The retraction r:* $\operatorname{Spv}A\to\operatorname{Spv}(A,I)$ *is spectral.*

(iv) *If v is a point of* $\operatorname{Spv}A$ *with $v(I)\neq\{0\}$ then also $r(v)(I)\neq\{0\}$.*

Proof. Let $f_0, \ldots, f_m, g_0, \ldots, g_n$ be elements of A. Then

$$\Big(\bigcap_{i=0,\ldots,m} \{v\in \operatorname{Spv} A \mid v(f_i)\leqq v(f_0)\neq 0\}\Big)\cap\Big(\bigcap_{j=0,\ldots,n} \{v\in \operatorname{Spv} A \mid v(g_j)\leqq v(g_0)\neq 0\}\Big)$$
$$= \bigcap_{\substack{i=0,\ldots,m\\ j=0,\ldots,n}} \{v\in \operatorname{Spv} A \mid v(f_i\, g_j)\leqq v(f_0\, g_0)\neq 0\},$$

and if

$I\subseteq\sqrt{(f_i\,|\,i=0,\ldots,m)}$ and $I\subseteq\sqrt{(g_j\,|\,j=0,\ldots,n)}$

then $I\subseteq\sqrt{(f_i\, g_j\,|\,i=0,\ldots,m, j=0,\ldots,n)}$. Hence the intersection of two elements of $\mathscr{R}$ is an element of $\mathscr{R}$. Next we prove the following two points.

(a) $\mathscr{R}$ is a basis of the topology of $\operatorname{Spv}(A, I)$.

(b) Let $g, f_1, \ldots, f_n$ be elements of A with $I\subseteq\sqrt{(f_1,\ldots,f_n)}$ and put $U=\{v\in\operatorname{Spv}(A,I)\,|\,v(f_1)\leqq v(g)\neq 0,\ldots,v(f_n)\leqq v(g)\neq 0\}$ and $W=\{v\in\operatorname{Spv} A\,|\,v(f_1)\leqq v(g)\neq 0,\ldots,v(f_n)\leqq v(g)\neq 0\}$. Then $W=r^{-1}(U)$.

Proof of (a). Let v be an element of $\operatorname{Spv}(A,I)$ and U neighbourhood of v in $\operatorname{Spv} A$. We choose $g_0,\ldots,g_n\in A$ such that $v\in W:=\{w\in\operatorname{Spv} A\,|\,w(g_i)\leqq w(g_0)\neq 0$ for $i=1,\ldots,n\}\subseteq U$. We distinguish two cases.

First case: $\Gamma_v=c\Gamma_v$.

Then there exists a $d\in A$ with $v(g_0\, d)\geqq 1$. Hence $v\in W':=\{w\in\operatorname{Spv} A\,|\,w(g_i\, d)\leqq w(g_0\, d)\neq 0$ for $i=1,\ldots,n$ and $w(1)\leqq w(g_0\, d)\neq 0\}\subseteq W$. We have $W'\cap\operatorname{Spv}(A,I)\in\mathscr{R}$.

Second case: $\Gamma_v\neq c\Gamma_v$.

Let $\{s_1,\ldots,s_m\}$ be a set of generators of the ideal J. By (2.5) there exists a $k\in\mathbb{N}$ with $v(s_i^k)\leqq v(g_0)$ for $i=1,\ldots,m$. Then $v\in W':=\{w\in\operatorname{Spv} A\,|\,w(g_i)\leqq w(g_0)\neq 0$ for $i=1,\ldots,n$ and $w(s_i^k)\leqq w(g_0)\neq 0$ for $i=1,\ldots,m\}\subseteq W$. We have $W'\cap\operatorname{Spv}(A,I)\in\mathscr{R}$.

Proof of (b). Since $U\subseteq W$ and since every point of $r^{-1}(U)$ specializes to a point of U we have $r^{-1}(U)\subseteq W$. Let $w\in W$ be given. We have to show $w\in r^{-1}(U)$, i.e. $r(w)\in W$. If $w(I)=\{0\}$ then $r(w)=w$. Now assume $w(I)\neq\{0\}$. Since $r(w)$ is a primary specialization of $w\in W$, we have $r(w)(f_i)\leqq r(w)(g)$ for $i=1,\ldots,n$. It remains to show $r(w)(g)\neq 0$. Assume to the contrary $r(w)(g)=0$. Then $r(w)(f_i)=0$ for $i=1,\ldots,n$ and hence $r(w)(i)=0$ for every $i\in I$ (since $I\subseteq\sqrt{(f_1,\ldots,f_n)}$). But by (2.4) and the definition of $c\Gamma_w(I)$ we have $w(I)\cap c\Gamma_w(I)\neq\varnothing$, i.e. there exists a $i\in I$ with $r(w)(i)\neq 0$, contradiction.

Let $\bar{\mathscr{R}}$ be the boolean algebra of subsets of $\operatorname{Spv}(A,I)$ generated by $\mathscr{R}$. Let T be the set $\operatorname{Spv}(A,I)$ equipped with the topology generated by $\bar{\mathscr{R}}$. By b) and (2.2) the mapping r: $(\operatorname{Spv} A)_{\text{cons}}\to T$ is continuous. Since $(\operatorname{Spv} A)_{\text{cons}}$ is quasi-compact (by (2.1.i)) and r is surjective, we obtain that T is quasi-compact. Now (a) and (2.1.vi) imply that $\operatorname{Spv}(A,I)$ is a spectral space and that every element of $\mathscr{R}$ is constructible in $\operatorname{Spv}(A,I)$. We conclude from (a) and (b) that r: $\operatorname{Spv} A\to\operatorname{Spv}(A,I)$ is spectral. (iv) follows from the proof of (b).

Remark 2.7 Let us consider the special case $I=A$. We have $c\Gamma_v(A)=c\Gamma_v$ for every valuation v of A. Hence $\operatorname{Spv}(A,A)=\{v\in\operatorname{Spv} A\,|\,\Gamma_v=c\Gamma_v\}$ is the set of all points of $\operatorname{Spv} A$ which have no proper primary specializations. Let r: $\operatorname{Spv} A\to\operatorname{Spv}(A,A)$ be the retraction. For every subset M of $\operatorname{Spv}(A,A)$, the set

$r^{-1}(M) \subseteq \operatorname{Spv} A$ is closed under primary specializations and primary generalizations. Hence $M \mapsto r^{-1}(M)$ is a bijection from the set of subsets of $\operatorname{Spv}(A, A)$ to the set of subsets of $\operatorname{Spv} A$ which are closed under primary specializatians and primary generalizations. Let M be a subset of $\operatorname{Spv}(A, A)$. By (2.1.v) and (2.6.iii) M is constructible (resp. pro-constructible) in $\operatorname{Spv}(A, A)$ if and only if $r^{-1}(M)$ is constructible (resp. pro-constructible) in $\operatorname{Spv} A$, and by (2.6.ii) M is constructible in $\operatorname{Spv}(A, A)$ if and only if M is a finite boolean combination of sets of the type $\{v \in \operatorname{Spv}(A, A) \mid v(f_1) \leqq v(g), \ldots, v(f_n) \leqq v(g)\}$, where $g, f_1, \ldots, f_n$ are elements of A with $A = f_1 A + \ldots + f_n A$.

3 Continuous valuations of f-adic rings

Let A be a f-adic ring. We call a valuation v of A *continuous* if for every $\gamma \in \Gamma_v$ there exists a neighbourhood U of 0 in A such that $v(u) < \gamma$ for every $u \in U$. We put $\operatorname{Cont} A = \{v \in \operatorname{Spv} A \mid v \text{ is continuous}\}$ and equip $\operatorname{Cont} A$ with the subspace topology of $\operatorname{Spv} A$. If $f\colon A \to B$ is a continuous ring homomorphism between f-adic rings and $v\colon B \to \Gamma \cup \{0\}$ a continuous valuation of B then $v \circ f\colon A \to \Gamma \cup \{0\}$ is a continuous valuation of A. Hence $\operatorname{Spv}(f)\colon \operatorname{Spv} B \to \operatorname{Spv} A$ restricts to a continuous mapping $\operatorname{Cont}(f)\colon \operatorname{Cont} B \to \operatorname{Cont} A$. If the topology of A is discrete then $\operatorname{Cont} A = \operatorname{Spv} A$. (Hence the functor Cont is a generalization of the functor Spv.) In this paragraph we want to study the functor Cont. Our first important result is

Theorem 3.1 *Let $A^{\circ\circ}$ be the set of topologically nilpotent elements of A. Then* $\operatorname{Cont} A = \{v \in \operatorname{Spv}(A, A^{\circ\circ} \cdot A) \mid v(a) < 1$ *for every* $a \in A^{\circ\circ}\}$.

Proof. If w is a continuous valuation of A then $w(a)$ is cofinal in Γ_w for every $a \in A^{\circ\circ}$, and hence by (2.5) $w \in \{v \in \operatorname{Spv}(A, A^{\circ\circ} \cdot A) \mid v(a) < 1$ for every $a \in A^{\circ\circ}\}$. Now let v be an element of $\operatorname{Spv}(A, A^{\circ\circ} \cdot A)$ with $v(a) < 1$ for every $a \in A^{\circ\circ}$. We have to show that v is continuous. For that we need

(1) $v(a)$ is cofinal in Γ_v for every $a \in A^{\circ\circ}$.

Proof of (1). If $\Gamma_v \neq c\Gamma_v$ then (1) follows from (2.5). So we assume $\Gamma_v = c\Gamma_v$. Let $a \in A^{\circ\circ}$ and $\gamma \in \Gamma_v$ be given. There exists a $t \in A$ with $v(t) \neq 0$ and $v(t)^{-1} \leqq \gamma$. We choose a $n \in \mathbb{N}$ with $ta^n \in A^{\circ\circ}$. Then $v(ta^n) < 1$ and hence $v(a)^n < \gamma$ which proves (1).

Let U be a subset of A and T a finite subset of U such that $\{U^n \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in A and $T \cdot U = U^2 \subseteq U$. Let $\gamma \in \Gamma_v$ be given. Since $U \subseteq A^{\circ\circ}$, we have $v(u) < 1$ for every $u \in U$. By (1) there exists a $n \in \mathbb{N}$ with $(\max\{v(t) \mid t \in T\})^n < \gamma$. Then $v(a) < \gamma$ for every $a \in T^n \cdot U = U^{n+1}$. Hence v is continuous.

Corollary 3.2 $\operatorname{Cont} A$ *is a closed (and hence pro-constructible) subset of the spectral space* $\operatorname{Spv}(A, A^{\circ\circ} \cdot A)$. *In particular*, $\operatorname{Cont} A$ *is a spectral space.*

For every $a \in A$ the set $\{v \in \operatorname{Cont} A \mid v(a) \leqq 1\}$ is constructible in $\operatorname{Cont} A$ (by (3.2), (2.1.iv) and (2.6.ii)). Let $\mathscr{F}_A$ be the set of pro-constructible subsets of $\operatorname{Cont} A$ which are intersections of sets of the form $\{v \in \operatorname{Cont} A \mid v(a) \leqq 1\}$ $(a \in A)$. Let $\mathscr{G}_A$ be the set subrings of A which are open and integrally closed in A. (For example, the ring A° of power-bounded elements of A is an element of $\mathscr{G}_A$.) There is a natural one-to-one correspondence between $\mathscr{F}_A$ and $\mathscr{G}_A$ which we describe in the following lemma.

Lemma 3.3 (i) *The mapping* $\sigma\colon \mathscr{G}_A \to \mathscr{F}_A$, $G \mapsto \{v \in \operatorname{Cont} A \mid v(g) \leqq 1$ *for all* $g \in G\}$ *is bijective. The inverse mapping is* $\tau\colon \mathscr{F}_A \to \mathscr{G}_A$, $F \mapsto \{a \in A \mid v(a) \leqq 1$ *for all* $v \in F\}$.

(ii) *Let* G *be an element of* $\mathscr{G}_A$ *with* $G \subseteq A^\circ$. *Then every point of* $\operatorname{Cont} A$ *is a secondary specialization of a point of* $\sigma(G)$. *In particular,* $\sigma(G)$ *is dense in* $\operatorname{Cont} A$.

(iii) *If* A *is a Tate ring and has a noetherian ring of definition then also the converse of* (ii) *holds: If* G *is an element of* $\mathscr{G}_A$ *such that* $\sigma(G)$ *is dense in* $\operatorname{Cont} A$ *then* $G \subseteq A^\circ$.

Proof. (i) Let G be an element of $\mathscr{G}_A$. We have to show $\tau(\sigma(G)) = G$. Obviously, $G \subseteq \tau(\sigma(G))$. Assume by way of contradiction that there exists a $a \in \tau(\sigma(G)) \setminus G$. Then we consider in the localization A_a the subring $G[a^{-1}]$. The element $a^{-1} \in G[a^{-1}]$ is not a unit of $G[a^{-1}]$ (since otherwise $a \in G[a^{-1}]$ which implies that a is integral over G and hence $a \in G$). Hence there exists a prime ideal $\mathfrak{p}$ of $G[a^{-1}]$ with $a^{-1} \in \mathfrak{p}$. Let $\mathfrak{q}$ be a minimal prime ideal of $G[a^{-1}]$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Choosing a valuation ring of $qf(G[a^{-1}]/\mathfrak{q})$ which dominates the local ring $(G[a^{-1}]/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}}$ we obtain a valuation s of $G[a^{-1}]$ with $\mathfrak{q} = \operatorname{supp}(s)$, $s(g) \leqq 1$ for all $g \in G$, and $s(x) < 1$ for all $x \in \mathfrak{p}$, in particular $s(a^{-1}) < 1$. Since there exists a prime ideal of A_a lying over $\mathfrak{q}$, there exists a valuation t of A_a lying over s. Put $u = t \mid A \in \operatorname{Spv} A$ and $v = u \mid c\Gamma_u \in \operatorname{Spv} A$. Then

(a) $v(a) > 1$
(b) $v(g) \leqq 1$ for all $g \in G$
(c) $v(x) < 1$ for all $x \in A^{\circ\circ}$
(d) $v \in \operatorname{Spv}(A, A^{\circ\circ} \cdot A)$.

Proof. (a) and (b) follow from the fact that $s(a^{-1}) < 1$ and $s(g) \leqq 1$ for all $g \in G$. Let $x \in A^{\circ\circ}$ be given. Since G is open, there exists a $n \in \mathbb{N}$ with $x^n a \in G$. Furthermore we remark that $x \in G$ (since G is open and integrally closed in A). Hence in $G[a^{-1}]$ we have $x^n = g a^{-1}$ with some $g \in G$. Since $a^{-1} \in \mathfrak{p}$, we obtain $x \in \mathfrak{p}$. Hence $s(x) < 1$ which implies $v(x) < 1$. By definition we have $v = u \mid c\Gamma_u$ and hence $v \in \operatorname{Spv}(A, A^{\circ\circ} \cdot A)$.

We conclude from (3.1) and (c), (d) that v is a continuous valuation, and with (b) we obtain $v \in \sigma(G)$. Now (a) implies $a \notin \tau(\sigma(G))$ which is a contradiction to our assumption $a \in \tau(\sigma(G))$.

(ii) Let v be a point of $\operatorname{Cont} A$. If $\operatorname{supp}(v)$ is open in A then $v/\Gamma_v \in \sigma(G)$. So we assume that $\operatorname{supp}(v)$ is not open in A. Then there exists a $a \in A^{\circ\circ}$ with $v(a) \neq 0$. Let H be the greatest convex subgroup of Γ_v with $v(a) \notin H$. We claim that $w := v/H \in \sigma(G)$. Obviously, w is a continuous valuation of A. Let g be an element of G. We have to show $w(g) \leqq 1$. Assume to the contrary that $w(g) > 1$. Since Γ_w has rank 1 and $w(a) \neq 0$, there exists a $n \in \mathbb{N}$ with $w(g^n a) > 1$. Since $a \in A^{\circ\circ}$ and $g \in A^\circ$, we have $g^n a \in A^{\circ\circ}$. Furthermore, w is continuous. Hence $w(g^n a) < 1$, contradiction.

(iii) Let G be an element of $\mathscr{G}_A$ such that $\sigma(G)$ is dense in $\operatorname{Cont} A$. We have to show $G \subseteq A^\circ$. Assume to the contrary that there exists a $g \in G \setminus A^\circ$. Then by (i) the set $T := \{v \in \sigma(A^\circ) \mid v(g) > 1\}$ is non empty. Let L be the set of the subsequent Lemma 3.4, and put $S = \{v \in L \mid v(g) > 1\}$. Then $T \subseteq S$ and hence $S \neq \varnothing$. Now (3.4) implies $S \cap (\operatorname{Cont} A)_{\max} \neq \varnothing$, i.e. there exists a maximal point w of $\operatorname{Cont} A$ with $w(g) > 1$. Since $w \notin \sigma(G)$ and w is maximal, w is not a specialization of a point of $\sigma(G)$. Hence by (2.1.iii) $\sigma(G)$ is not dense in $\operatorname{Cont} A$, contradiction.

Lemma 3.4 *Let A be a Tate ring which has a noetherian ring of definition. Put $L=\{v\in\operatorname{Spv}A\,|\,v(a)\leqq 1$ for all $a\in A^\circ$ and $v(a)<1$ for all $a\in A^{\circ\circ}\}$ and $(\operatorname{Cont}A)_{\max}=\{v\in\operatorname{Cont}A\,|\,v$ is maximal in $\operatorname{Cont}A\}$. Then L is the closure of $(\operatorname{Cont}A)_{\max}$ in the constructible topology of* Spv A.

Proof. First we observe

(1) $(\operatorname{Cont}A)_{\max}=\{v\in\operatorname{Cont}A\,|\,\operatorname{rank}(\Gamma_v)=1\}$. (Indeed, every $v\in\operatorname{Cont}A$ has no proper primary specialization (since A is a Tate ring). Hence (2.3) implies that every specialization in Cont A is secondary. Now (1) is obvious.) (1) and the computation in the proof of (3.3.ii) show $(\operatorname{Cont}A)_{\max}\subseteq L$. Let T be a constructible subset of Spv A with $T\cap L\neq\varnothing$. We have to show that $T\cap(\operatorname{Cont}A)_{\max}\neq\varnothing$. We may assume $T=\{w\in\operatorname{Spv}A\,|\,w(a_i)\leqq w(b_i)$ for $i=1,\dots,m$ and $w(c_i)<w(d_i)$ for $i=1,\dots,n\}$ with a_i, b_i, c_i, $d_i\in A$. Let B be a noetherian ring of definition of A and let I be an ideal of definition of B. We choose an element $t\in T\cap L$. Let $K=qf(A/\operatorname{supp}(t))$ be the residue field of A at supp(t) and let f: $A\to K$ be the natural ring homomorphism. We may assume that $f(b_i)\neq 0$ for $i=1,\dots,k$ and $f(b_i)=0$ for $i=k+1,\dots,m$. Let C be the subring $f(B)\left[\frac{f(a_i)}{f(b_i)},\ i=1,\dots,k;\ \frac{f(c_i)}{f(d_i)},\ i=1,\dots,n\right]$ of K. The valuation ring $A(t)$ of K contains C, and the maximal ideal $\mathfrak{m}$ of $A(t)$ contains $f(I)\cup\left\{\frac{f(c_i)}{f(d_i)}\,\middle|\,i=1,\dots,n\right\}$. The prime ideal $\mathfrak{m}\cap C$ of C is not the zero ideal, since $f(I)\neq\{0\}$. Hence by [5, 0.6.5.8] there exists a (discrete) rank 1 valuation ring D of K which dominates the local ring $C_{\mathfrak{m}\cap C}$. Let v be the valuation of A with $\operatorname{supp}(v)=\operatorname{supp}(t)$ and $A(v)=D$. Then $v\in(\operatorname{Cont}A)_{\max}$ (by (1)) and $v\in T$.

In many applications it is necessary to work with a subspace $F\subseteq\operatorname{Cont}A$ which is an element of $\mathscr{F}_A$ instead of the whole space Cont A (cf. [8]). But of course we should not lose too much information by the transition from Cont A to F, in particular F should be dense in Cont A. So (3.3) motivates the following definition.

Definition. (i) A subring of a f-adic ring A which is open and integrally closed in A and contained in A° is called a *ring of integral elements* of A.

(ii) An *affinoid ring* A is a pair $A=(A^\rhd, A^+)$ where $A^\rhd$ is a f-adic ring and A^+ is a ring of integral elements of $A^\rhd$. An affinoid ring A is called adic (resp. Tate, resp. complete) if $A^\rhd$ is adic (resp. Tate, resp. complete). A ring homomorphism f: $A\to B$ between affinoid rings is a ring homomorphism f: $A^\rhd\to B^\rhd$ with $f(A^+)\subseteq B^+$. The ring homomorphism f: $A\to B$ is called continuous (resp. adic) if f: $A^\rhd\to B^\rhd$ is continuous (resp. adic).

(iii) For an affinoid ring A, Spa A denotes the subspace $\{v\in\operatorname{Cont}A^\rhd\,|\,v(a)\leqq 1$ for all $a\in A^+\}$ of Cont $A^\rhd$. If f: $A\to B$ is a continuous ring homomorphism between affinoid rings then $\operatorname{Cont}(f)$: $\operatorname{Cont}B^\rhd\to\operatorname{Cont}A^\rhd$ indudes per restriction a continuous mapping $\operatorname{Spa}(f)$: Spa $B\to$ Spa A.

If A is a f-adic ring then the integral closure B of the subring $\mathbb{Z}\cdot 1+A^{\circ\circ}$ in A is the smallest ring of integral elements of A and $\operatorname{Cont}A=\operatorname{Spa}(A,B)$.

Let A be an affinoid ring. The subsets of Spa A of the form

$$R\left(\frac{T_1}{s_1},\dots,\frac{T_n}{s_n}\right):=\bigcap_{i=1}^{n}\{v\in\operatorname{Spa}A\,|\,v(t)\leqq v(s_i)\neq 0\ \text{ for all }\ t\in T_i\},$$

where $s_1, \dots, s_n$ are elements of $A^{\rhd}$ and $T_1, \dots, T_n$ are finite subsets of $A^{\rhd}$ such that $T_i \cdot A^{\rhd}$ is open in $A^{\rhd}$ for $i=1, \dots, n$ are called *rational*.

If T is a finite subset of $A^{\rhd}$ then $T \cdot A^{\rhd}$ is open in $A^{\rhd}$ if and only if $(A^{\rhd})^{\circ\circ} \cdot A^{\rhd} \subset \sqrt{T \cdot A^{\rhd}}$. By (2.6.ii) and (3.1) Spa A is a pro-constructible subset of $\operatorname{Spv}(A^{\rhd}, (A^{\rhd})^{\circ\circ} \cdot A^{\rhd})$. Hence (2.1.iv) and (2.6) imply our first main theorem.

Theorem 3.5 (i) Spa A *is a spectral space.*

(ii) *The rational subsets form a basis of* Spa A *and every rational subset is constructible in* Spa A.

(iii) *To every rational subset U of* Spa A *there exist an element s of $A^{\rhd}$ and a finite subset T of $A^{\rhd}$ such that $T \cdot A^{\rhd}$ is open in $A^{\rhd}$ and $U = R\left(\frac{T}{s}\right)$.*

Notice that if A is Tate then the rational subsets of Spa A are the sets of the form $\{v \in \operatorname{Spa} A \mid v(t_i) \leqq v(s) \text{ for } i=1, \dots, n\}$, where $s, t_1, \dots, t_n$ are elements of $A^{\rhd}$ with $A^{\rhd} = t_1 A^{\rhd} + \dots + t_n A^{\rhd}$.

By (3.5.ii) a subset L of Spa A is constructible if and only if L is a finite boolean combination of rational subsets.

A point $x \in \operatorname{Spa} A$ is called *analytic* if the support $\operatorname{supp}(x) \subseteq A^{\rhd}$ is not open in $A^{\rhd}$. (Note that every valuation of $A^{\rhd}$ with open support is continuous.) We put $(\operatorname{Spa} A)_a := \{x \in \operatorname{Spa} A \mid x \text{ is analytic}\}$ and $(\operatorname{Spa} A)_{na} := \operatorname{Spa} A \setminus (\operatorname{Spa} A)_a$. The sets $(\operatorname{Spa} A)_a$ and $(\operatorname{Spa} A)_{na}$ are constructible in Spa A. Indeed, if T is a finite subset of $(A^{\rhd})^{\circ\circ}$ such that $T \cdot A^{\rhd}$ is open in $A^{\rhd}$ then $(\operatorname{Spa} A)_a = \{x \in \operatorname{Spa} A \mid x(t) \neq 0$ for some $t \in T\} = \bigcup_{t \in T} R\left(\frac{T}{t}\right)$. We see that $(\operatorname{Spa} A)_a$ is open in Spa A. If A is Tate then $\operatorname{Spa} A = (\operatorname{Spa} A)_a$, and if the topology of $A^{\rhd}$ is discrete then $\operatorname{Spa} A = (\operatorname{Spa} A)_{na}$. In $(\operatorname{Spa} A)_a$ there are no proper primary specializations, and hence by (2.3) every specialization in $(\operatorname{Spa} A)_a$ is a secondary specialization. For every $x \in (\operatorname{Spa} A)_a$ we have $\operatorname{rank}(\Gamma_x) \geqq 1$, and $\operatorname{rank}(\Gamma_x) = 1$ if and only if x is a maximal point of $(\operatorname{Spa} A)_a$.

The Zariski-spectrum Spec B of a ring B is empty if and only if B is the zero ring. Similarly we have for Spa A

Proposition 3.6 *Let A be an affinoid ring. Then*

(i) Spa $A = \varnothing$ *if and only if the Hausdorff ring $A^{\rhd}/\overline{\{0\}}$ associated with $A^{\rhd}$ is the zero ring.*

(ii) $(\operatorname{Spa} A)_a = \varnothing$ *if and only if the topology of the Hausdorff ring $A^{\rhd}/\overline{\{0\}}$ is discrete.*

Proof. (ii) Assume that $(\operatorname{Spa} A)_a = \varnothing$. Let B be a ring of definition of $A^{\rhd}$ and I a finitely generated ideal of definition of B. Then we have

(1) Let $\mathfrak{p}$ be a prime ideal of B such that there exists a prime ideal $\mathfrak{q}$ of B with $\mathfrak{p} \subseteq \mathfrak{q}$ and $I \subseteq \mathfrak{q}$. Then $I \subseteq \mathfrak{p}$.

Proof. Assume that $I \nsubseteq \mathfrak{p}$. Let u be a valuation of B such that $\mathfrak{p} = \operatorname{supp}(u)$ and $B(u)$ dominates the local ring $(B/\mathfrak{p})_{\mathfrak{q}/\mathfrak{p}}$. Let r: $\operatorname{Spv} B \to \operatorname{Spv}(B, I)$ be the retraction from (2.6.iii). Then $r(u)$ is a continuous valuation of B with $I \nsubseteq \operatorname{supp}(r(u))$ (by (3.1) and (2.6.iv)). By the subsequent Lemma 3.7 there exists a continuous valuation v of $A^{\rhd}$ with $r(u) = v \mid B$. The secondary generalization w of v with $\operatorname{rank}(\Gamma_w) = 1$ is an element of $(\operatorname{Spa} A)_a$ (cf. proof of (3.3.ii)), contradiction.

Let $\varphi\colon B \to C$ be the localization of B by the multiplicative system $S=1+I$. Since in Spec C every prime ideal specializes to a prime ideal containing $\varphi(I)\,C$, (1) implies that $\varphi(I)\,C$ is contained in every prime ideal of C. Therefore there exists a $n\in\mathbb{N}$ with $\varphi(I^n)\,C=\{0\}$, i.e. there exists a $i\in I$ with $(1+i)\,I^n=\{0\}$ in B. Then $I^n=I^k$ for every $k\geqq n$, and hence the topology of $A^{\triangleright}/\overline{\{0\}}$ is discrete.

(i) Assume Spa $A=\varnothing$. Then the ideal $\overline{\{0\}}$ is open in $A^{\triangleright}$ (by (ii)). Hence every trivial valuation v of $A^{\triangleright}$ with $\overline{\{0\}}\subseteq\operatorname{supp}(v)$ is an element of Spa A. This shows that no prime ideal of $A^{\triangleright}$ contains $\overline{\{0\}}$, i.e. $\overline{\{0\}}=A^{\triangleright}$.

Lemma 3.7 *Let B be an open subring of a* $\mathfrak{f}$*-adic ring A. Let f:* Spec $A\to$ Spec B *be the morphism of schemes induced by the inclusion $B\subseteq A$. Let T be the closed subset* $\{\mathfrak{p}\in\operatorname{Spec} B\,|\,\mathfrak{p}$ *is open*$\}$ *of* Spec B. *Then* $f^{-1}(T)=\{\mathfrak{p}\in\operatorname{Spec} A\,|\,\mathfrak{p}$ *is open*$\}$ *and the restriction* Spec $A\setminus f^{-1}(T)\to$ Spec $B\setminus T$ *of f is an isomorphism.*

Proof. Let $\mathfrak{p}\in\operatorname{Spec} B\setminus T$ be given. We choose a $s\in B^{\circ\circ}$ with $s\notin\mathfrak{p}$. For every $a\in A$ there exists a $n\in\mathbb{N}$ with $s^n a\in B$ (since B is open in A). Hence the ring homomorphism $B_s\to A_s$ is bijective.

Proposition 3.8 *Let $f\colon A\to B$ be a continuous ring homomorphism between affinoid rings, and let $g\colon X:=$* Spa $B\to Y:=$ Spa A *be the mapping induced by f. Then*

(i) $g(X_{na})\subseteq Y_{na}$.

(ii) *If f is adic then $g(X_a)\subseteq Y_a$.*

(iii) *If B is complete and $g(X_a)\subseteq Y_a$ then f is adic.*

(iv) *If f is adic then g is spectral (more precisely, the preimage under g of a rational subset is rational).*

Proof. (iii) Assume that B is complete and that f is not adic. We will show that $g(X_a)\nsubseteq Y_a$. We choose rings of definition A_0 and B_0 of $A^{\triangleright}$ and $B^{\triangleright}$ and finitely generated ideals of definition I_A and I_B of A_0 and B_0 with $f(A_0)\subseteq B_0$ and $f(I_A)\subseteq I_B$. Since f is not adic, there exists a prime ideal $\mathfrak{p}$ of B_0 with $f(I_A)\subseteq\mathfrak{p}$ and $I_B\nsubseteq\mathfrak{p}$. Since B_0 is complete in the I_B-adic topology, there exists a prime ideal $\mathfrak{q}$ of B_0 with $\mathfrak{p}\subseteq\mathfrak{q}$ and $I_B\subseteq\mathfrak{q}$ [3, III.2.13 Lemma 3]. Now we repeat the arguments of the proof of (3.6.ii). Let u be a valuation of B_0 such that $\operatorname{supp}(u)=\mathfrak{p}$ and $B_0(u)$ dominates the local ring $(B_0/\mathfrak{p})_{\mathfrak{q}/\mathfrak{p}}$, and let r: Spv $B_0\to\operatorname{Spv}(B_0, I_B)$ be the retraction from (2.6.iii). Then $r(u)$ is a continuous valuation of B_0 with $I_B\nsubseteq\operatorname{supp}(r(u))$. By (3.7) there exists a continuous valuation v *of* $B^{\triangleright}$ with $r(u)=v\,|\,B_0$. Let w be the secondary generalization of v with rank $(\Gamma_w)=1$. Then $w\in X_a$ and $g(w)\notin Y_a$.

(iv) Let s be an element of $A^{\triangleright}$ and T a finite subset of $A^{\triangleright}$ such that $T\cdot A^{\triangleright}$ is open in $A^{\triangleright}$. If f is adic then $f(T)\cdot B^{\triangleright}$ is open in $B^{\triangleright}$, and hence $g^{-1}\left(R\left(\frac{T}{s}\right)\right)$ is the rational subset $R\left(\frac{f(T)}{f(s)}\right)$ of Spa B.

Let $A=(B, C)$ be an affinoid ring. Then the completion $\hat{C}$ of C is a ring of integral elements of the completion $\hat{B}$ of B. We call the affinoid ring $\hat{A}:=(\hat{B}, \hat{C})$ the completion of A.

Proposition 3.9 *The canonical mapping g:* Spa $\hat{A}\to$ Spa A *is a homeomorphism, and it maps rational subsets to rational subsets.*

In order to prove (3.9) we need the following two lemmata.

Lemma 3.10 *Let A be a complete affinoid ring and let $s, t_1, \ldots, t_n$ be elements of $A^{\rhd}$ such that the ideal $I = t_1 A^{\rhd} + \ldots + t_n A^{\rhd}$ is open in $A^{\rhd}$. Then there exists a neighbourhood U of 0 in $A^{\rhd}$ such that, for $s' \in s + U$, $t'_1 \in t_1 + U, \ldots, t'_n \in t_n + U$, the ideal $I' = t'_1 A^{\rhd} + \ldots + t'_n A^{\rhd}$ is open in $A^{\rhd}$ and $R\left(\frac{t_1, \ldots, t_n}{s}\right) = R\left(\frac{t'_1, \ldots, t'_n}{s'}\right)$.*

Proof. Let B be a ring of definition of $A^{\rhd}$. Let $r_1, \ldots, r_m$ be elements of $B \cap I$ such that $J := r_1 B + \ldots + r_m B$ is open. By [3, III.2.10 Corollary 3] there exists a neighbourhood V of 0 in B such that $J = r'_1 B + \ldots + r'_m B$ for every $r'_1 \in r_1 + V, \ldots, r'_m \in r_m + V$. (That is the only point where we use that A is complete.) Hence there exists a neighbourhood U' of 0 in $A^{\rhd}$ such that $t'_1 A^{\rhd} + \ldots + t'_n A^{\rhd}$ is open in $A^{\rhd}$ for every $t'_1 \in t_1 + U', \ldots, t'_n \in t_n + U'$.

We put $t_0 := s$. For every $i \in \{0, \ldots, n\}$ let R_i be the rational subset $R\left(\frac{t_0, \ldots, t_n}{t_i}\right)$. Then R_i is quasi-compact (by (3.5.ii) and (2.1.i)) and $x(t_i) \neq 0$ for every $x \in R_i$. Hence by the subsequent Lemma 3.11 there exists a neighbourhood U'' of 0 in $A^{\rhd}$ such that $x(u) < x(t_i)$ for every $u \in U''$, $i \in \{0, \ldots, n\}$ and $x \in R_i$. We will show that (3.10) holds for $U = U' \cap U'' \cap (A^{\rhd})^{\circ\circ}$.

First we show $R_0 \subseteq R\left(\frac{t'_1, \ldots, t'_n}{t'_0}\right)$. Let $x \in R_0$ be given. Since $t'_i - t_i \in U''$ for $i = 0, \ldots, n$, we have $x(t'_i - t_i) < x(t_0)$ for $i = 0, \ldots, n$. This implies for every $i = 1, \ldots, n$

$$x(t'_i) = x(t_i + (t'_i - t_i)) \leqq \max\{x(t_i), x(t'_i - t_i)\} \leqq x(t_0) = x(t_0 + (t'_0 - t_0)) = x(t'_0).$$

Hence $x \in R\left(\frac{t'_1, \ldots, t'_n}{t'_0}\right)$.

Now let x be an element of $\operatorname{Spa} A$ with $x \notin R_0$. We have to show $x \notin R\left(\frac{t'_1, \ldots, t'_n}{t'_0}\right)$. First we consider the case that $x(t_i) = 0$ for $i = 0, \ldots, n$. Then $\operatorname{supp}(x)$ is open (since the ideal I is open). Hence $t'_0 - t_0 \in \operatorname{supp}(x)$ (since $t'_0 - t_0 \in (A^{\rhd})^{\circ\circ}$) which implies $t'_0 \in \operatorname{supp}(x)$. Hence $x \notin R\left(\frac{t'_1, \ldots, t'_n}{t'_0}\right)$.

Now assume that $x(t_i) \neq 0$ for some $i \in \{0, \ldots, n\}$. We choose a $j \in \{0, \ldots, n\}$ with $x(t_j) = \max\{x(t_0), \ldots, x(t_n)\}$. We have $x(t_0) < x(t_j)$, since otherwise we would have $x \in R_0$. As $t'_i - t_i \in U''$ for $i = 0, \ldots, n$ and $x \in R_j$, we have $x(t'_i - t_i) < x(t_j)$ for $i = 0, \ldots, n$. Then

$$x(t'_0) = x(t_0 + (t'_0 - t_0)) \leqq \max\{x(t_0), x(t'_0 - t_0)\} < x(t_j) = x(t_j + (t'_j - t_j)) = x(t'_j).$$

Hence $x \notin R\left(\frac{t'_1, \ldots, t'_n}{t'_0}\right)$.

Lemma 3.11 *Let A be an affinoid ring, X a quasi-compact subset of $\operatorname{Spa} A$ and s an element of $A^{\rhd}$ with $x(s) \neq 0$ for every $x \in X$. Then there exists a neighbourhood U of 0 in $A^{\rhd}$ such that $x(u) < x(s)$ for every $x \in X$, $u \in U$.*

Proof. Let T be a finite subset of $(A^{\rhd})^{\circ\circ}$ such that $T \cdot (A^{\rhd})^{\circ\circ}$ is open. For every $n \in \mathbb{N}$ put $X_n = \{x \in \operatorname{Spa} A \mid x(t) \leqq x(s) \neq 0 \text{ for every } t \in T^n\}$. Then X_n is open in

Spa A, and $X \subseteq \bigcup_{n \in \mathbb{N}} X_n$. Hence $X \subseteq X_m$ for some $m \in \mathbb{N}$. Put $U = T^m \cdot (A^{\triangleright})^{\circ\circ}$. Then U is a neighbourhood of 0 in $A^{\triangleright}$ and $x(u) < x(s)$ for every $x \in X, u \in U$.

Now we prove (3.9). Clearly g is bijective. Let U be a rational subset of Spa $\hat{A}$. We have to show that $g(U)$ is rational in Spa A. Let i: $A^{\triangleright} \to (A^{\triangleright})^{\wedge}$ be the natural mapping. By (3.10) there exist an element $s \in A^{\triangleright}$ and a finite subset T of $A^{\triangleright}$ with $U = R\left(\frac{i(T)}{i(s)}\right)$. Since U is quasi-compact by ((3.5.ii) and (2.1.i)) and since $x(i(s)) \neq 0$ for every $x \in U$, there exists a neighbourhood G of 0 in $A^{\triangleright}$ such that $x(i(g)) \leqq x(i(s))$ for every $x \in U$ and every $g \in G$ (by (3.11)). Let D be a finite subset of G such that $D \cdot A^{\triangleright}$ is open. Then we have the rational subset $V := R\left(\frac{T \cup D}{s}\right)$ of Spa A and $V = g(U)$.

4 Tate rings of topologically finite type over a field

In this section we consider a special type of Tate rings. Let k be a field which is complete with respect to a rank 1 valuation $|\ |$: $k \to \mathbb{R}^{>} \cup \{0\}$. We put $k\langle X_1, \ldots, X_n \rangle = \{\sum a_\nu X^\nu \in k[\![X_1, \ldots, X_n]\!] \mid (a_\nu)_{\nu \in \mathbb{N}_0^n}$ is a zero sequence in $k\}$ and equip $k\langle X_1, \ldots, X_n \rangle$ with the topology induced by the norm $\|\ \|$ with $\|\sum_{\nu \in \mathbb{N}_0^n} a_\nu X^\nu\| = \max\{|a_\nu| \mid \nu \in \mathbb{N}_0^n\}$. Then $k\langle X_1, \ldots, X_n \rangle$ is a complete Tate ring. We call a complete topological k-algebra A a Tate algebra over k (more precisely, a Tate ring of topologically finite type over k) if there exists a continuous, open and surjective k-algebra homomorphism $k\langle X_1, \ldots, X_n \rangle \to A$ for some $n \in \mathbb{N}_0$ (cf. [13, 4, 2]). Every Tate algebra over k is noetherian [2, 6.1.1 Proposition 3] and every k-algebra homomorphism between Tate algebras over k is continuous [2, 6.1.3 Theorem 1].

In this paragraph we show that, for every Tate algebra A over k, the category of sheaves of the rigid analytic variety Sp A is canonically isomorphic to the category of sheaves of the topological space Spa(A, A°). We give two proofs. In the first proof we use the subsequent Theorem 4.1 which is useful also for other applications (cf. [9]). In order to prove (4.1) we use the model theoretic result that the theory of algebraically closed fields with non trivial valuation-divisibility relation has elimination of quantifiers. For the second proof we use some standard facts on Tate algebras.

We fix a Tate algebra A over k. Let Max A be the set of all maximal ideals of A. For every $x \in$ Max A, the residue field A/x is finite over k [2, 6.1.2 Corollary 3]. Hence the valuation $|\ |$ of k extends uniquely to a valuation $|\ |_x$ of A with support x. Since x is closed in A [2, 6.1.1 Proposition 3], it is easily seen that $|\ |_x$ is a continuous valuation of A, and then even $|\ |_x \in$ Spa(A, A°). So we have an injective mapping Max $A \to$ Spa(A, A°), $x \mapsto |\ |_x$. We consider Max A as a subset of Spa(A, A°) via this mapping. We put $L_A = \{v \in \operatorname{Spv} A \mid v(a) \leqq 1$ for all $a \in A^\circ$ and $v(a) < 1$ for all $a \in A^{\circ\circ}\}$. Then

$$\operatorname{Max} A \subseteq \operatorname{Spa}(A, A^\circ) \subseteq L_A \subseteq \operatorname{Spv} A.$$

(*Remark.* If r: Spv $A \to$ Spv(A, A) is the retraction from (2.6.iii) then $L_A = r^{-1}(\operatorname{Spa}(A, A^\circ))$). Every k-algebra homomorphism $A \to B$ from A to a Tate algebra B over k induces mappings $\mathrm{Spv}\, B \to \mathrm{Spv}\, A$, $L_B \to L_A$, $\mathrm{Spa}(B, B^\circ) \to \mathrm{Spa}(A, A^\circ)$, $\mathrm{Max}\, B \to \mathrm{Max}\, A$.

Theorem 4.1 *L_A is the closure of* $\mathrm{Max}\, A$ *in the constructible topology of* $\mathrm{Spv}\, A$.

Proof. First we show by induction on n that (4.1) is true for $A = k(n) := k\langle X_1, \ldots, X_n\rangle$. For $A = k(0) = k$ we have $L_A = \mathrm{Max}\, A$. Assume that (4.1) is true for $A = k(n)$. We will show that (4.1) is true for $A = k(n+1)$. Let T be a non empty constructible subset of $L_{k(n+1)}$. We have to show that $T \cap \mathrm{Max}\, k(n+1) \neq \emptyset$. We may assume

$$T = \{v \in L_{k(n+1)} \mid v(a_i) \leqq v(b_i) \text{ and } v(c_i) < v(d_i) \text{ for } i = 1, \ldots, m\}$$

with a_i, b_i, c_i, $d_i \in k(n+1)$. By the Weierstrass preparation theorem [2, 5.2.2 and 5.2.4] there exists a k-algebra automorphism σ of $k(n+1)$ such that $\sigma(a_i) = \mathfrak{a}_i \cdot A_i, \ldots, \sigma(d_i) = \mathfrak{d}_i \cdot D_i$, where $\mathfrak{a}_i, \ldots, \mathfrak{d}_i$ are units of $k(n+1)$ of the form $1+x$ with $x \in k(n+1)^{\circ\circ}$ (cf. [2, 5.1.3 Proposition 1]) and $A_i, \ldots, D_i \in k(n)[X_{n+1}]$. Let f: $\mathrm{Spv}\, k(n+1) \to \mathrm{Spv}\, k(n+1)$ be the mapping induced by σ. By definition of $L_{k(n+1)}$ we have $v(1+x) = 1$ and $v(X_{n+1}) \leqq 1$ for every $x \in k(n+1)^{\circ\circ}$ and every $v \in L_{k(n+1)}$. Hence

$$f^{-1}(T) = \{v \in L_{k(n+1)} \mid v(A_i) \leqq v(B_i) \text{ and } v(C_i) < v(D_i) \text{ for } i = 1, \ldots, m \text{ and } v(X_{n+1}) \leqq 1\}.$$

Let g: $L_{k(n+1)} \to \mathrm{Spv}\, k(n)[X_{n+1}]$ and h: $\mathrm{Spv}\, k(n)[X_{n+1}] \to \mathrm{Spv}\, k(n)$ be the mappings induced by the inclusions $k(n)[X_{n+1}] \subseteq k(n+1)$ and $k(n) \subseteq k(n)[X_{n+1}]$. We put

$$S = \{v \in \mathrm{Spv}\, k(n)[X_{n+1}] \mid v(A_i) \leqq v(B_i) \text{ and } v(C_i) < v(D_i) \text{ for } i = 1, \ldots, m \text{ and } v(X_{n+1}) \leqq 1\}.$$

Since $T \neq \emptyset$ and $g(f^{-1}(T)) \subseteq S$ and $\mathrm{im}(h \circ g) \subseteq L_{k(n)}$, we have $h(S) \cap L_{k(n)} \neq \emptyset$. The model theoretic result that the theory of algebraically closed fields with non trivial valuation-divisibility relation has elimination of quantifiers [11, 4.17] implies that $h(S)$ is a constructible subset of $\mathrm{Spv}\, k(n)$. So by induction hypothesis we have $h(S) \cap \mathrm{Max}\, k(n) \neq \emptyset$. We choose a $x \in h(S) \cap \mathrm{Max}\, k(n)$. Applying again that the theory of algebraically closed fields with non trivial valuation-divisibility relation has elimination of quantifiers, we obtain that there exists a $y \in \mathrm{Spv}\, k(n)[X_{n+1}]$ such that $y \in S$, $x = h(y)$ and $qf(k(n)[X_{n+1}]/\mathrm{supp}(y))$ is algebraic over $qf(k(n)/\mathrm{supp}(x))$. Then $\mathrm{supp}(y)$ is a maximal ideal of $k(n)[X_{n+1}]$ and y is continuous with respect to the subspace topology of $k(n+1)$ on $k(n)[X_{n+1}]$ (here we use that $y(X_{n+1}) \leqq 1$). Hence y extends to a continuous valuation z of the completion $(k(n)[X_{n+1}])^\wedge = k(n+1)$ with $z \in \mathrm{Max}\, k(n+1)$. Since $f^{-1}(T) = g^{-1}(S)$, we obtain $z \in f^{-1}(T) \cap \mathrm{Max}\, k(n+1) = f^{-1}(T \cap \mathrm{Max}\, k(n+1))$. Hence $T \cap \mathrm{Max}\, k(n+1) \neq \emptyset$.

Now let A be an arbitrary Tate algebra over k. We choose a surjective k-algebra homomorphism p: $k(n) \to A$ for some $n \in \mathbb{N}_0$. Let q: $(\mathrm{Spv}\, A)_{\mathrm{cons}} \to (\mathrm{Spv}\, k(n))_{\mathrm{cons}}$ be the mapping induced by p. We know that $L_{k(n)}$ is the closure of $\mathrm{Max}\, k(n)$ in $(\mathrm{Spv}\, k(n))_{\mathrm{cons}}$. Since q is open, we obtain that $q^{-1}(L_{k(n)})$ is the closure of $q^{-1}(\mathrm{Max}\, k(n)) = \mathrm{Max}\, A$ in $(\mathrm{Spv}\, A)_{\mathrm{cons}}$. But $\mathrm{Max}\, A \subseteq L_A \subseteq q^{-1}(L_{k(n)})$ and L_A is closed in $(\mathrm{Spv}\, A)_{\mathrm{cons}}$. Hence L_A is the closure of $\mathrm{Max}\, A$ in $(\mathrm{Spv}\, A)_{\mathrm{cons}}$.

Corollary 4.2 *Max A is dense in the constructible topology of* Spa(A, A°).

Proof. Let T be a non empty constructible subset of Spa(A, A°). By (3.5.ii) T is a finite boolean combination of rational subsets of Spa(A, A°). Hence there exists a constructible subset M of Spv A with $T = M \cap \mathrm{Spa}(A, A^\circ)$. Consequently $M \cap L_A \neq \varnothing$ (since $\mathrm{Spa}(A, A^\circ) \subseteq L_A$). Then by (4.1) $T \cap \mathrm{Max}\, A = M \cap \mathrm{Max}\, A \neq \varnothing$.

Corollary 4.3 *Let X_1, X_2 be constuctible subsets of* Spa(A, A°) *with $X_1 \cap \mathrm{Max}\, A = X_2 \cap \mathrm{Max}\, A$. Then $X_1 = X_2$.*

Proof. $X := (X_1 \setminus X_2) \cup (X_2 \setminus X_1)$ is a constructible subset of Spa(A, A°) with $X \cap \mathrm{Max}\, A = \varnothing$. Then by (4.2) $X = \varnothing$, i.e. $X_1 = X_2$.

A subset R of Max A is called rational if there exist $f_1, \ldots, f_n$, $g \in A$ such that $R = \{x \in \mathrm{Max}\, A \mid |f_i|_x \leqq |g|_x \text{ for } i = 1, \ldots, n\}$ and $A = f_1 A + \ldots + f_n A$ [4, III.1.1]. The intersection of two rational subsets of Max A is rational. If U is a rational subset of Spa(A, A°) then $U \cap \mathrm{Max}\, A$ is a rational subset of Max A. (4.3) implies that $U \mapsto U \cap \mathrm{Max}\, A$ is a bijection from the set of rational subsets of Spa(A, A°) to the set of rational subsets of Max A.

In rigid analytic geometry one associates with A a Grothendieck topology $\mathcal{T}_A$ [4, III.2.1]: The objects of the category $\mathcal{T}_A$ are the rational subsets of Max A. If U, V are rational subsets of Max A then $\mathrm{Hom}(U, V) = \varnothing$ if $U \nsubseteq V$ and $|\mathrm{Hom}(U, V)| = 1$ if $U \subseteq V$. A family $(U_i)_{i \in I}$ of rational subsets of Max A is a covering of a rational subset U if there exists a finite subset J of I with $U = \bigcup_{i \in I} U_i = \bigcup_{j \in J} U_j$. From (3.5) and (4.3) we can deduce

Corollary 4.4 *The category of sheaves of the Grothendieck topology $\mathcal{T}_A$ is canonically isomorphic to the category of sheaves of the topological space* Spa(A, A°).

Proof. Let F be a sheaf on $\mathcal{T}_A$. For every open subset U of Spa(A, A°) we put $\tilde{F}(U) = \varprojlim_V F(V \cap \mathrm{Max}\, A)$, where the projective limit is taken over all rational subsets V of Spa(A, A°) with $V \subseteq U$. By (3.5.ii) and (2.1.i) every rational subset of Spa(A, A°) is quasi-compact. Then we can conclude from [5, 0.3.2.2] that $U \mapsto \tilde{F}(U)$ is a sheaf on Spa(A, A°). Thus we have a functor i: $S \to \tilde{S}$ from the category S of sheaves on $\mathcal{T}_A$ to the category $\tilde{S}$ of sheaves on Spa(A, A°). Now let $\tilde{G}$ be a sheaf on Spa(A, A°). For every rational subset U of Max A let $\tilde{U}$ be the rational subset of Spa(A, A°) with $U = \tilde{U} \cap \mathrm{Max}\, A$. Then $U \mapsto G(U) := \tilde{G}(\tilde{U})$ is a sheaf on $\mathcal{T}_A$ (by (4.3)). So we have a functor j: $\tilde{S} \to S$ which is quasi-inverse to i.

A set $\mathcal{F}$ of rational subsets of Max A is called a prime filter if (i) $\mathrm{Max}\, A \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$; (ii) if X_1, $X_2 \in \mathcal{F}$ then $X_1 \cap X_2 \in \mathcal{F}$; (iii) if $X_1 \in \mathcal{F}$ and if X_2 is a rational subset of Max A with $X_1 \subseteq X_2$ then $X_2 \in \mathcal{F}$; (iv) if $X_1, \ldots, X_n$ are rational subsets of Max A with $X_1 \cup \ldots \cup X_n \in \mathcal{F}$ then $X_i \in \mathcal{F}$ for some $i \in \{1, \ldots, n\}$. Let $(\mathrm{Max}\, A)^\sim$ denote the set of prime filters of rational subsets of Max A. We equip $(\mathrm{Max}\, A)^\sim$ with the topology generated by the sets $\{\mathcal{F} \in (\mathrm{Max}\, A)^\sim \mid F \in \mathcal{F}\}$ with F a rational subset of Max A.

Corollary 4.5 *If x is a point of* Spa(A, A°) *then $s(x) := \{U \cap \mathrm{Max}\, A \mid U$ rational subset of* Spa(A, A°) *with $x \in U\}$ is a prime filter of rational subsets of* Max A. *The mapping s:* $\mathrm{Spa}(A, A^\circ) \to (\mathrm{Max}\, A)^\sim$, *$x \mapsto s(x)$ is a homeomorphism.*

Proof. By (4.3), $s(x)\in(\operatorname{Max} A)^\sim$. Let x, y be different points of $\operatorname{Spa}(A, A^\circ)$. Since $\operatorname{Spa}(A, A^\circ)$ is a T_0-space and the set of rational subsets of $\operatorname{Spa}(A, A^\circ)$ form a basis (3.5.ii), there exists a rational subset U of $\operatorname{Spa}(A, A^\circ)$ such that $x\in U$, $y\notin U$ or $x\notin U$, $y\in U$. Hence $s(x)\neq s(y)$.

Let $\mathscr{F}\in(\operatorname{Max} A)^\sim$ be given. Put $\mathscr{W}=\mathscr{F}\cup\{(\operatorname{Max} A)\setminus R \mid R$ rational subset of $\operatorname{Max} A$ with $R\notin\mathscr{F}\}$, and for every $W\in\mathscr{W}$ let $W^\sim$ be the constructible subset of $\operatorname{Spa}(A, A^\circ)$ with $W^\sim\cap\operatorname{Max} A=W$. For every finite subset $\mathscr{E}$ of $\mathscr{W}$ we have $\bigcap_{W\in\mathscr{E}} W^\sim\neq\varnothing$ (since $\bigcap_{W\in\mathscr{E}} W\neq\varnothing$). As $\operatorname{Spa}(A, A^\circ)_{\mathrm{cons}}$ is compact ((2.1.i) and (3.5.i)), we obtain that $D:=\bigcap_{W\in\mathscr{W}} W^\sim$ is non empty. For every $x\in D$ we have $s(x)=\mathscr{F}$. Hence s is bijective. Since the rational subsets of $\operatorname{Spa}(A, A^\circ)$ form a basis, s is a homeomorphism.

Remark 4.6 We define a relation $\leq$ on $(\operatorname{Max} A)^\sim$ by $\mathscr{F}_1\leq\mathscr{F}_2$ iff $\mathscr{F}_1\subseteq\mathscr{F}_2$. Then $\mathscr{F}_1\leq\mathscr{F}_2$ if and only if $s^{-1}(\mathscr{F}_1)$ is a specialization of $s^{-1}(\mathscr{F}_2)$ in $\operatorname{Spa}(A, A^\circ)$. Hence the mapping s induces a bijection from the set $\operatorname{Spa}(A, A^\circ)_{\max}$ of maximal points of $\operatorname{Spa}(A, A^\circ)$ to the set of maximal elements of $(\operatorname{Max} A)^\sim$. Since $\operatorname{Spa}(A, A^\circ)_{\max}$ consists of the continuous rank 1 valuations of A, we obtain the following result of van der Put [12, 1.3.3 Corollary]: There is a natural one-to-one correspondence between the continuous rank 1 valuations of A and the maximal prime filters of rational subsets of $\operatorname{Max} A$.

Remark 4.7 We deduced (4.2)–(4.5) as consequences of (3.5,i, ii) and (4.1). In this remark we give new proofs of (4.2)–(4.5) without using (3.5.i, ii) and (4.1). Furthermore, we will give a new proof of (3.5.i, ii) in case that the affinoid ring A is of the form $A=(B, B^\circ)$ where B is a Tate algebra over k.

Let A be a Tate algebra over k, and let $\mathcal{O}$ be the structure sheaf of the rigid analytic variety associated with A [4, III.2.1]. If v is a continuous valuation of A and U is a rational subset of $\operatorname{Max} A$ then v can be extended in at most one way to a continuous valuation of $\mathcal{O}(U)$, since $\mathcal{O}(U)$ is the completion of a localization of A [2, 6.1.4].

(4.7.1) *Let $g_i, f_{i1}, \ldots, f_{in(i)}$ $(i=1, \ldots, k)$ be elements of A such that $A=f_{i1}A+\ldots+f_{in(i)}A$ for $i=1, \ldots, k$, let U be the rational subset $\bigcap_{i=1}^{k}\{x\in \operatorname{Max} A \mid |f_{ij}|_x\leq|g_i|_x$ for $j=1, \ldots, n(i)\}$ of* $\operatorname{Max} A$, *and let v be an element of* $\operatorname{Spa}(A, A^\circ)$. *Then the following conditions are equivalent.*

(i) *v extends to a continuous valuation w of $\mathcal{O}(U)$ such that $w(e)\leq 1$ for every $e\in\mathcal{O}(U)^\circ$ (i.e., v lies in the image of the natural mapping* $\operatorname{Spa}(\mathcal{O}(U), \mathcal{O}(U)^\circ)\to\operatorname{Spa}(A, A^\circ)$).

(ii) *$v(f_{ij})\leq v(g_i)$ for $i=1, \ldots, k, j=1, \ldots, n(i)$.*

Proof. g_i is a unit in $\mathcal{O}(U)$, and $\dfrac{f_{ij}}{g_i}$ is power-bounded in $\mathcal{O}(U)$. Hence (i) implies (ii). Now we assume (ii), and we will show that (i) holds. Let B be the valuation ring of $L:=qf(A/\operatorname{supp}(v))$ such that B has rank 1 and contains the valuation ring $A(v)\subseteq L$ associated with v. Then $B\cap k=k^\circ$, and hence the valuation $|\ |$ of k extends to a valuation $|\ |: L\to\mathbb{R}$ with $B=\{x\in L \mid |x|\leq 1\}$. Let $(K, |\ |)$ be the completion of $(L, |\ |)$. Then $(K, |\ |)$ is a k-Banach algebra, and the natural ring homomorphism $\varphi\colon A\to K$ is continuous. Furthermore, $\varphi(g_i)$ is a unit of K and $\frac{\varphi(f_{ij})}{\varphi(g_i)}$ is power-bounded in K. Hence by the universal property [2, 6.1.4, Proposition 3] there is a continuous A-algebra homomorphism $\varphi'\colon \mathcal{O}(U)\to K$. This implies that v extends to a continuous valuation w of $\mathcal{O}(U)$. We have $w\left(\frac{f_{ij}}{g_i}\right)\leqq 1$. Let C be the integral closure of $A^\circ\left[\frac{f_{ij}}{g_i}\,\middle|\, i=1,\dots,k, j=1,\dots,n(i)\right]$ in $\mathcal{O}(U)$. Then $w(c)\leqq 1$ for every $c\in C$, and then, since w is continuous, $w(c)\leqq 1$ for every element c of the closure $\bar{C}$ of C in $\mathcal{O}(U)$. One can show $\bar{C}=\mathcal{O}(U)^\circ$ (cf. [8, 4.4]). Hence (i) is satisfied.

(4.7.2) *Let $\mathcal{F}\in(\operatorname{Max} A)^\sim$ be a prime filter of rational subsets of* Max A. *We put* $\mathfrak{p}_{\mathcal{F}}:=\{a\in A\mid$for every $F\in\mathcal{F}$ there exists a $x\in F$ with $a(x)=0\}=\{a\in A\mid$for every $e\in k^*$ *there exists a* $F\in\mathcal{F}$ *with* $|a|_x\leqq|e|_x$ *for every* $x\in F\}$. *Then there exists (up to equivalence) a unique valuation $v_{\mathcal{F}}$ of A such that, for all a, $b\in A$, $v_{\mathcal{F}}(a)\leqq v_{\mathcal{F}}(b)$ if and only if $a\in\mathfrak{p}_{\mathcal{F}}$ or there exists a $F\in\mathcal{F}$ with $|a|_x\leqq|b|_x$ for every $x\in F$. We have $v_{\mathcal{F}}\in\operatorname{Spa}(A, A^\circ)$.*

Proof. Let $|_{\mathcal{F}}$ be the binary relation of A with $b|_{\mathcal{F}}a$ if and only if $a\in\mathfrak{p}_{\mathcal{F}}$ or there exists a $F\in\mathcal{F}$ with $|a|_x\leqq|b|_x$ for every $x\in F$. Then $|_{\mathcal{F}}$ satisfies the conditions (1)–(6) of the proof of (2.2). This says that there exists (up to equivalence) a unique valuation $v_{\mathcal{F}}$ of A such that, for all a, $b\in A$, $v_{\mathcal{F}}(a)\leqq v_{\mathcal{F}}(b)$ if and only if $a\in\mathfrak{p}_{\mathcal{F}}$ or there exists a $F\in\mathcal{F}$ with $|a|_x\leqq|b|_x$ for every $x\in F$. It is easily seen that $v_{\mathcal{F}}(a)\leqq 1$ for all $a\in A^\circ$ [4, II.5.5], $\mathfrak{p}_{\mathcal{F}}=\operatorname{supp}(v_{\mathcal{F}})$ and, for every $a\in A\setminus\operatorname{supp}(v_{\mathcal{F}})$, there exist e_1, $e_2\in k^*$ with $v_{\mathcal{F}}(e_1)\leqq v_{\mathcal{F}}(a)\leqq v_{\mathcal{F}}(e_2)$. Hence $v_{\mathcal{F}}\in\operatorname{Spa}(A, A^\circ)$.

We call a subset X of $(\operatorname{Max} A)^\sim$ rational if there exists a rational subset U of Max A such that $X=U^\sim:=\{\mathcal{F}\in(\operatorname{Max} A)^\sim\mid U\in\mathcal{F}\}$. By the definition of the topology of $(\operatorname{Max} A)^\sim$, the rational subsets of $(\operatorname{Max} A)^\sim$ form a basis of the topology of $(\operatorname{Max} A)^\sim$. For every $x\in\operatorname{Max} A$, the set $j(x)$ of all rational subsets U of Max A with $x\in U$ is a prime filter. Via the injection $j\colon \operatorname{Max} A\to(\operatorname{Max} A)^\sim$, $x\mapsto j(x)$ we consider Max A as a subset of $(\operatorname{Max} A)^\sim$. With (4.7.1) and (4.7.2) we obtain

(4.7.3) *For every $v\in\operatorname{Spa}(A, A^\circ)$, the set $d(v)$ of all rational subsets U of* Max A *such that v extends to a continuous valuation w of $\mathcal{O}(U)$ with $w(e)\leqq 1$ for every $e\in\mathcal{O}(U)^\circ$ is a prime filter of rational subsets of* Max A. *The mapping $d\colon\operatorname{Spa}(A, A^\circ)\to(\operatorname{Max} A)^\sim$, $v\mapsto d(v)$ is a homeomorphism. If X is a rational subset of $(\operatorname{Max} A)^\sim$ then $d^{-1}(X)$ is a rational subset of $\operatorname{Spa}(A, A^\circ)$, and the mapping $X\mapsto d^{-1}(X)$ is a bijection from the set of rational subsets of $(\operatorname{Max} A)^\sim$ onto the set of rational subsets of $\operatorname{Spa}(A, A^\circ)$. Furthermore, $d^{-1}(\operatorname{Max} A)=\operatorname{Max} A$.*

Proof. Let $v\in\operatorname{Spa}(A, A^\circ)$ be given. We show that $d(v)$ is a prime filter. Obviously, $d(v)$ satisfies the condition (i) and (iii) of the definition of a prime filter. Condition (ii) follows from (4.7.1). Let U be an element of $d(v)$ and $U_1,\dots,U_n$ rational subsets of Max A with $U=U_1\cup\dots\cup U_n$. We have to show that $U_i\in d(v)$ for some $i\in\{1,\dots,n\}$. By [4, III.2.5] there exist $f_1,\dots,f_m\in\mathcal{O}(U)$ such that $\mathcal{O}(U)=f_1\,\mathcal{O}(U)+\dots+f_m\,\mathcal{O}(U)$ and, for every $i\in\{1,\dots,m\}$, the rational subset $V_i:=\{x\in U\mid |f_j|_x\leqq|f_i|_x$ for $j=1,\dots,m\}$ of Max A is contained in some U_l. Let $w\in\operatorname{Spa}(\mathcal{O}(U), \mathcal{O}(U)^\circ)$ be the continuous extension of v to $\mathcal{O}(U)$. We choose a $r\in\{1,\dots,m\}$ with $w(f_i)\leqq w(f_r)$ for $i=1,\dots,m$. Then (4.7.1) implies $V_r\in d(v)$, and hence $U_i\in d(v)$ for some $i\in\{1,\dots,n\}$.

Let $\mathscr{F}$ be a filter of rational subsets of Max A, and let $v_{\mathscr{F}} \in \operatorname{Spa}(A, A^\circ)$ be the corresponding valuation as defined in (4.7.2). Then by (4.7.1), $\mathscr{F} = d(v_{\mathscr{F}})$. Hence d is surjective.

The rational subsets of $\operatorname{Spa}(A, A^\circ)$ form a basis of the topology of $\operatorname{Spa}(A, A^\circ)$. Indeed, let $v \in \operatorname{Spa}(A, A^\circ)$ be contained in an open subset $U := \{x \in \operatorname{Spa}(A, A^\circ) \mid x(a) \leqq x(b) \neq 0\}$ $(a, b \in A)$ of $\operatorname{Spa}(A, A^\circ)$. Since v is continuous, there exists a $c \in k^*$ with $v(c) \leqq v(b)$. Then $V := \{x \in \operatorname{Spa}(A, A^\circ) \mid x(a) \leqq x(b) \neq 0$ and $x(c) \leqq x(b) \neq 0\}$ is a rational subset of $\operatorname{Spa}(A, A^\circ)$ with $v \in V \subseteq U$.

Let $g, f_1, \ldots, f_n$ be elements of A with $A = f_1 A + \ldots + f_n A$. If we put $U = \{x \in \operatorname{Max} A \mid |f_i|_x \leqq |g|_x$ for $i = 1, \ldots, n\}$ and $V = \{v \in \operatorname{Spa}(A, A^\circ) \mid v(f_i) \leqq v(g)$ for $i = 1, \ldots, n\}$ then $d^{-1}(U^\sim) = V$ (by (4.7.1)). Since the topology of $\operatorname{Spa}(A, A^\circ)$ is T_0 and the rational subsets form a basis, we can conclude that d: $\operatorname{Spa}(A, A^\circ) \to (\operatorname{Max} A)^\sim$ is a homeomorphism and $X \mapsto d^{-1}(X)$ is a bijection from the set of rational subsets of $(\operatorname{Max} A)^\sim$ to the set of rational subsets of $\operatorname{Spa}(A, A^\circ)$.

(4.7.4) *Let $\mathscr{B}$ be the boolean algebra of subsets of* $(\operatorname{Max} A)^\sim$ *which is generated by the rational subsets of* $(\operatorname{Max} A)^\sim$, *and let $\mathscr{T}$ be the topology of* $(\operatorname{Max} A)^\sim$ *generated by $\mathscr{B}$. Then* $((\operatorname{Max} A)^\sim, \mathscr{T})$ *is compact, and $\mathscr{B}$ is the set of subsets of* $(\operatorname{Max} A)^\sim$ *which are open and closed in $\mathscr{T}$.* Max A *is dense in* $((\operatorname{Max} A)^\sim, \mathscr{T})$.

Proof. Let S be the set of rational subsets of Max A, and let $\mathscr{P}(S)$ be the power set of S. Then $(\operatorname{Max} A)^\sim$ is a subset of $\mathscr{P}(S)$. We equip $\{0, 1\}$ with the discrete topology and $\mathscr{P}(S) = \{0, 1\}^S$ with the product topology. Then $\mathscr{P}(S)$ is compact, and $((\operatorname{Max} A)^\sim, \mathscr{T})$ is a closed subspace of $\mathscr{P}(S)$. □

The mapping $X \mapsto X \cap \operatorname{Max} A$ is a bijection from the set of rational subsets of $(\operatorname{Max} A)^\sim$ to the set of rational subsets of Max A. The inverse mapping is $U \mapsto U^\sim$. A family $(U_i)_{i \in I}$ of rational subsets of Max A is a covering of a rational subset U of Max A in the Grothendieck topology of Max A if and only if $U^\sim = \bigcup_{i \in I} U_i^\sim$, since $U^\sim$ is quasi-compact in the topology of $(\operatorname{Max} A)^\sim$ (by (4.7.4)). Hence the category of sheaves of the Grothendieck topology of Max A is canonically equivalent to the category of sheaves of the topological space $(\operatorname{Max} A)^\sim$ (cf. [5, 0.3.2.2]). Now (4.7.3) implies (4.4).

By (2.1.vi) and (4.7.4), $(\operatorname{Max} A)^\sim$ is a spectral space and $\mathscr{B}$ is the set of constructible subsets of $(\operatorname{Max} A)^\sim$. Hence with (4.7.3) we obtain (3.5.i, ii) for the affinoid ring (A, A°). By (4.7.4), Max A is dense in the constructible topology of $(\operatorname{Max} A)^\sim$. Then again (4.7.3) implies (4.2).

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